

# USING CARRY-TRUNCATED ADDITION TO ANALYZE ADD-ROTATE-XOR HASH ALGORITHMS

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**ABSTRACT.** We introduce a truncated addition operation on pairs of  $N$ -bit binary numbers that interpolates between ordinary addition mod  $2^N$  and bitwise addition in  $(\mathbb{Z}/2\mathbb{Z})^N$ . We use truncated addition to analyze hash functions that are built from the bit operations add, rotate, and xor, such as `Blake`, `Skein`, and `Cubehash`. Any ARX algorithm can be approximated by replacing ordinary addition with truncated addition, and we define a metric on such algorithms which we call the **sensitivity**. This metric measures the smallest approximation agreeing with the full algorithm a statistically useful portion of the time (we use 0.1%). Because truncated addition greatly reduces the complexity of the non-linear operation in ARX algorithms, the approximated algorithms are more susceptible to both collision and pre-image attacks, and we outline a potential collision attack explicitly. We particularize some of these observations to the `Skein` hash function.

## 1. INTRODUCTION

This paper is concerned with a family of hash algorithms that are defined in terms of addition mod  $2^N$  (denoted  $+$ ), bitwise rotation, and exclusive or (denoted  $\oplus$ ) which is equivalent to bitwise addition mod 2. Such algorithms are referred to as ARX algorithms.

The non-linearity of ARX algorithms over  $(\mathbb{Z}/2\mathbb{Z})^N$  relies exclusively on the addition mod  $2^N$  component. As in base 10, we can perform addition on each digit and keep a carry value for each position to record overflows. In base 2, we will observe that carrying occurs frequently and so the addition-with-carrying operation is indeed highly non-linear and we note that computers are designed to compute this type of non-linearity efficiently.

In this work, we replace ordinary addition mod  $2^N$  with a series of approximations that converge to actual addition. These approximations arise from truncating the number of carry values that we record. The zeroth approximation is addition with no carries which corresponds to the exclusive-or operation. The first approximation is bitwise addition plus a single carry term for each bit; namely, we look back a single bit for carry terms and do not “carry our carries.” The second approximation involves looking back two bits for carry information, and so on. The surprising fact is that for 64-bit binary numbers, the fourth approximation and the actual sum coincide a statistically useful percentage of the time. The eighth approximation coincides with ordinary addition more than 90 percent of the time.

In light of this, it is natural to consider replacing instances of ordinary addition in an ARX algorithm by the simpler truncated addition operation. We describe a polynomial encoding for hash algorithms that can in principle be used to find collisions and preimages for the algorithm with truncated addition. Although neither attack is currently practical, we show that replacing ordinary addition by truncated addition dramatically reduces the degree of these polynomials, which should facilitate their analysis. When collisions exist in the version using truncated addition and the algorithm using truncated addition agrees with the usual algorithm sufficiently often, then one obtains collisions in the full algorithm with a significant nonzero probability.

We also use this setting to describe a new metric that measures the strength of ARX hash algorithms. This metric can be described as the number of carry bits that must be used before we can find cases where the full algorithm and its approximation using truncated addition agree a statistically useful percent of the

time. We measure this using a computer implementation of the algorithm and a random search through 10 million inputs. This metric is found to agree with the popular wisdom, based on factors such as the speed of hashing, that Cubehash160+16/32+160-256 is stronger than the ARX algorithms that were final round candidates for the SHA-3 competition. This algorithm requires 13 bits of carrying before matches can be found. In contrast, we were able to find 29 cases of agreement per 10,000 random inputs using only 9 bits of carrying for the algorithm Skein. This means that it suffices to attack the 9-truncated approximation rather than the full addition version of Skein as its approximation coincides a sufficient percent of the time.

The main technique in this paper, replacing addition with truncated addition, has been used as part of cryptographic attacks in the past. In [5], a series of approximations for the hash algorithm Salsa20/8 (a reduced round version of the full Salsa algorithm) are shown to possess the same bias in differential probabilities as the full algorithm. As the full key is not necessary to trace backwards for the approximate algorithm, this differential bias can be used to distinguish key conjectures that are good candidates for the approximate to the true key. Using a combination of second and third order approximation (two or three carry bits are recorded, but no others), the authors are able to show that a key can be found in a better than exhaustive search.

Here, we use truncated addition to define a new (and concrete) method to compare the robustness of different ARX algorithms. As part of this comparison, each algorithm is assigned an approximation of sufficient complexity that any cryptographic attack can be applied to the approximations with statistically significant results for the full algorithm. We also provide a direct combinatorial proof of the exact probability that truncated addition and ordinary addition will produce the same result, a significant improvement to approximations such as “the  $d^{\text{th}}$  order term may be ignored with probability  $1 - 2^{-d}$ ” currently in the literature [5].

In Section 2 we describe our truncated addition operation in detail. In Section 3 we explain how to encode a hash algorithm as a system of polynomial equations. Section 4 gives some empirical data about hash algorithms from the NIST competition [6]. In Section 5 we give some suggestions for future research regarding the algorithm Skein. A short conclusion follows in Section 6.

## 2. AN APPROXIMATION TO ADDITION BY TRUNCATED CARRIES

Fix an integer  $N$ . In our applications  $N = 32$  or  $N = 64$ , and we represent integers in binary notation using  $N$  bits. For example,  $x = \sum_{i=0}^{N-1} x_i 2^i$  has binary digits  $x_i \in \{0, 1\}$ . We will sometimes write these digits as an array  $[x_{N-1}, x_{N-2}, \dots, x_1, x_0]$  with the least significant bit in the rightmost position.

**Definition 2.1.** Let  $x = \sum_{i=0}^{N-1} x_i 2^i$  and  $y = \sum_{i=0}^{N-1} y_i 2^i$ . We can then view  $x$  and  $y$  as elements of  $\mathbb{Z}/(2^N \mathbb{Z})$  and  $(\mathbb{Z}/2\mathbb{Z})^N$  simultaneously. Here,  $\mathbb{Z}/(2^N \mathbb{Z})$  represents the group of integers with addition mod  $2^N$ , while  $(\mathbb{Z}/2\mathbb{Z})^N$  represents bitstrings of length  $N$  under componentwise addition mod 2. We denote the ordinary addition of these integers in  $\mathbb{Z}/(2^N \mathbb{Z})$  by  $x + y$ . We denote the bitwise addition of these integers

$$\sum_{i=0}^{N-1} ((x_i + y_i) \bmod 2) 2^i$$

in  $(\mathbb{Z}/2\mathbb{Z})^N$  by  $x \oplus y$ .

To relate these operations, we introduce the **carry array**  $c(x, y) = \sum_{i=1}^{N-1} c_i(x, y) 2^i$ , where

$$c_i(x, y) = \begin{cases} 1 & \text{if } x_{i-1} + y_{i-1} + c_{i-1}(x, y) \in \{2, 3\} \\ 0 & \text{otherwise.} \end{cases}$$

Then the usual addition algorithm using carries yields

$$x + y = x \oplus y \oplus c(x, y).$$

Observe that  $c_0(x, y)$  is always 0 by definition. If  $x_{N-1} = 1$  and  $y_{N-1} = 1$ , then we would generate a carry at the  $N$ th position, but  $2^N = 0$  in  $\mathbb{Z}/(2^N\mathbb{Z})$  so we omit this.

**Lemma 2.2.** *We have that  $c_i(x, y) = 1$  if and only if there exists  $j < i$  such that  $(x_j, y_j) = (1, 1)$  and for all  $j < k < i$ , we have  $x_k + y_k = 1$ .*

*Proof.* It follows from the definitions that

$$c_i(x, y) = \begin{cases} c_{i-1}(x, y) & \text{if } c_{i-1}(x, y) = 1 \text{ and } x_{i-1} + y_{i-1} \in \{1, 2\} \\ 0 & \text{if } c_{i-1}(x, y) = 1 \text{ and } x_{i-1} + y_{i-1} = 0 \\ c_{i-1}(x, y) & \text{if } c_{i-1}(x, y) = 0 \text{ and } x_{i-1} + y_{i-1} \in \{0, 1\} \\ 1 & \text{if } c_{i-1}(x, y) = 0 \text{ and } x_{i-1} + y_{i-1} = 2. \end{cases}$$

Hence, strings of carrying are started by a  $(x_j, y_j) = (1, 1)$  pair, continued by  $(0, 1)$ ,  $(1, 0)$  and  $(1, 1)$  pairs, and stopped by a  $(0, 0)$  pair. If there are multiple  $(1, 1)$  pairs prior to position  $i$ , we choose the pair with the greatest position  $j$  so that  $(x_k, y_k) \in \{(0, 1), (1, 0)\}$  for all  $j < k < i$  by construction.  $\square$

Observe that in the worst case, we might have to look back  $N - 1$  positions to decide whether a carry exists at the most significant position. We now define a version of addition based on a carry array that uses the information from at most  $m$  prior positions.

**Definition 2.3.** Let  $c_i^{(m)}(x, y)$  be 1 if there exists  $i - m \leq j < i$  such that  $(x_j, y_j) = (1, 1)$  and for all  $j < k < i$  we have  $x_k + y_k = 1$ . We then define the  **$m$ -truncated addition** of  $x$  and  $y$  to be

$$x +_m y := x \oplus y \oplus c^{(m)}.$$

where  $c^{(m)} = \sum_{i=1}^{N-1} c_i^{(m)} 2^i$ .

Observe that  $x +_0 y = x \oplus y$  and  $x +_{(N-1)} y = x + y$  so truncated addition generalizes and interpolates between these operations.

**Example 2.4.** If  $N = 4$  then

$$\begin{array}{rcccc} & 1 & 0 & 0 & 1 \\ +_3 & 1 & 0 & 1 & 1 \\ \hline & 0 & 1 & 0 & 0 \end{array} \qquad \begin{array}{rcccc} & 1 & 0 & 0 & 1 \\ +_1 & 1 & 0 & 1 & 1 \\ \hline & 0 & 0 & 0 & 0 \end{array}$$

represents  $9 + 11 = 20$  which is equivalent to  $4 \pmod{2^4}$ , and  $9 +_1 11 = 0$ , respectively. In the first case where  $m = N - 1 = 3$ , the carry array is  $c^{(3)} = [0, 1, 1, 0]$ . In the second case where  $m = 1$ , the 1-truncated carry array is  $c^{(1)} = [0, 0, 1, 0]$ . We see that  $c_2^{(1)} = 0$  since there is no  $(1, 1)$  pair lying within  $m = 1$  positions prior to position  $i = 2$ . On the other hand,  $c_2^{(3)} = 1$  since there does exist a  $(1, 1)$  pair lying within  $m = 3$  positions prior to position  $i = 2$ .

**Proposition 2.5.** *We have  $x + y = x +_m y$  if and only if the sequence  $\{x_i + y_i\}_{i=0}^{N-2}$  does not contain a 2 directly followed by a contiguous subsequence of  $m$  1's as  $i$  runs from 0 to  $N - 2$ .*

*Proof.* This follows by comparing Lemma 2.2 and Definition 2.3.  $\square$

We are now in a position to determine the probability that  $x +_m y$  agrees with  $x + y$ . Recall that a **ternary string** is one in which each digit is 0, 1 or 2.

$m$	$N = 32\text{-bit}$	$N = 64\text{-bit}$
4	63.62771 %	37.10136 %
5	80.94266 %	62.31794 %
6	90.49360 %	79.59719 %
7	95.36429 %	89.50263 %
8	97.76392 %	94.73115 %
9	98.92764 %	97.38680 %
10	99.48763 %	98.71143 %
11	99.75591 %	99.36646 %
12	99.88404 %	99.68900 %
13	99.94507 %	99.84747 %
14	99.97406 %	99.92525 %
15	99.98779 %	99.96338 %
16	99.99428 %	99.98207 %

TABLE 1. Probability of  $x +_m y = x + y$ 

**Lemma 2.6.** *Let  $P(m)$  be the ternary string  $1^m 2 = 11 \cdots 12$ . Let  $p_m(i)$  be the probability that in a bitwise sum of uniformly chosen binary strings (of any length  $\geq m + 1$ ), the rightmost instance of  $P(m)$  as a consecutive substring ends at position  $i$ . Here, we label the positions from right to left, starting from 0. Let  $a_m(j)$  be the probability that a bitwise sum of uniformly chosen binary strings of length  $j$  does not contain  $P(m)$  as a consecutive substring. Then we have the system*

$$(2.1) \quad a_m(j) = 1 - \sum_{i=0}^{(j-1)-m} p_m(i)$$

$$(2.2) \quad p_m(i) = \left(\frac{1}{2}\right)^m \left(\frac{1}{4}\right) a_m(i)$$

that can be solved explicitly for  $a_m(N - 1)$ .

*Proof.* Every instance of  $P(m)$  in a ternary string of length  $j$  must end at some position, and each such event is independent, so Equation (2.1) represents the probability that no instances of  $P(m)$  occur. Equation (2.2) gives the probability that in the bitwise sum of two uniformly chosen binary strings, the rightmost  $i$  positions avoid  $P(m)$ , the next position is a 2 (this occurs with probability  $1/4$ ), the  $m$  subsequent positions are 1's (these each occur with probability  $1/2$ ), and the remaining positions are all unrestricted (so contribute probability 1).  $\square$

**Corollary 2.7.** *The probability  $\pi_m(N)$  that  $x +_m y = x + y$  where  $x$  and  $y$  are uniformly chosen  $N$ -bit integers is  $a_m(N - 1)$ . Some typical values of  $\pi_m(N)$  are illustrated in Table 1.*

*Proof.* This follows from Proposition 2.5 and Lemma 2.6.  $\square$

### 3. A POLYNOMIAL ENCODING AND METRICS FOR ARX ALGORITHMS

In this section, we consider encoding an ARX hash algorithm by a system of polynomial functions over  $\mathbb{F}_2$ , the 2-element field. Here, we mean that the domain, range, and ring of coefficients of these polynomials should all be  $\mathbb{F}_2$ . We will see that replacing instances of  $+$  by  $+_m$  reduces the degree of

these polynomials, which facilitates analysis of the hash algorithm. At the same time, Table 1 gives some evidence that making this replacement will not change the output of the hash function too often.

Observe that our  $N$ -bit arrays have an action of the symmetric group  $\mathcal{S}_N$  of permutations on  $N$  letters given by permuting the entries of arrays. In particular, this action allows us to achieve the bitwise rotation operation. We denote this action by  $\sigma \cdot [x_{N-1}, \dots, x_0]$  for  $\sigma \in \mathcal{S}_N$ .

**Proposition 3.1.** *Consider two  $N$ -bit arrays  $x = [x_{N-1}, \dots, x_1, x_0]$  and  $y = [y_{N-1}, \dots, y_1, y_0]$ , and let  $\sigma \in \mathcal{S}_N$ . There exist polynomial functions in  $\mathbb{F}_2[x_0, x_1, \dots, x_{N-1}, y_0, y_1, \dots, y_{N-1}]$  whose evaluation is equal to the  $i$ th bit of  $x \oplus y$ ,  $x + y$  and  $\sigma \cdot x$ , respectively. Explicitly, we have*

- The  $i$ th bit of  $\sigma \cdot [x_{N-1}, \dots, x_1, x_0]$  is  $x_{\sigma(i)}$ .
- The  $i$ th bit of  $[x_{N-1}, \dots, x_1, x_0] \oplus [y_{N-1}, \dots, y_1, y_0]$  is  $x_i + y_i$ .
- The  $i$ th bit of  $[x_{N-1}, \dots, x_1, x_0] +_m [y_{N-1}, \dots, y_1, y_0]$  is

$$(x_i + y_i) + \sum_{k=1}^{\min(i,m)} (x_{i-k}y_{i-k}) \prod_{j=i-k+1}^{i-1} (x_j + y_j).$$

*Proof.* The first two formulas are straightforward. The last formula follows from Definition 2.3.  $\square$

**Example 3.2.** The addition of two 4-bit numbers  $[x_3, x_2, x_1, x_0] + [y_3, y_2, y_1, y_0]$  can be represented by the polynomials

$$[(x_3 + y_3) + (x_2y_2) + (x_1y_1)(x_2 + y_2) + (x_0y_0)(x_1 + y_1)(x_2 + y_2), \\ (x_2 + y_2) + (x_1y_1) + (x_0y_0)(x_1 + y_1), (x_1 + y_1) + (x_0y_0), x_0 + y_0]$$

with maximum degree 4. If we use 2-truncated addition instead, then we obtain

$$[(x_3 + y_3) + (x_2y_2) + (x_1y_1)(x_2 + y_2), (x_2 + y_2) + (x_1y_1) + (x_0y_0)(x_1 + y_1), \\ (x_1 + y_1) + (x_0y_0), x_0 + y_0],$$

which has maximum degree 3.

We consider an **APX hash function** to be any finite composition of the operations  $+$ ,  $\oplus$ , and any permutation of the bits in an array. To find a collision for such a hash algorithm, it is helpful to have a message that is at least as long as the output. We therefore let  $n$  be the maximum number of bits in the input (including both the message as well as any key derived from the message), output, or internal state.

Let  $\bar{x}_i$  be variables representing the bits of input to the hash, so each  $\bar{x}_i \in \{0, 1\}$  for  $0 \leq i \leq n-1$ . We include variable bits for the key if it is derived from the message. We then use Proposition 3.1 to build polynomials  $\bar{y}_i \in \mathbb{F}_2[\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{n-1}]$  that represent the  $i$ th bit of output from the APX hash function. We can encode multiple rounds of a sub-algorithm by iterating the functions we obtain, taking the  $\bar{y}_i$  expressed in terms of the  $\bar{x}_i$  and using them as input.

If we do this for two sets of inputs  $\bar{x}_i$  and  $\bar{x}'_i$ , say, then collisions correspond to nontrivial solutions of the system of polynomial equations

$$\{\bar{y}_i(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{n-1}) = \bar{y}_i(\bar{x}'_0, \bar{x}'_1, \dots, \bar{x}'_{n-1})\}_{i=0}^{n-1}.$$

Similarly, if we let  $\bar{z}_i$  be variables corresponding to the output of a hash, then a preimage for the output  $(\bar{z}_0, \dots, \bar{z}_{n-1})$  corresponds to a solution of the system of polynomial equations

$$\{\bar{y}_i(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{n-1}) = \bar{z}_i\}_{i=0}^{n-1}.$$

These systems each have  $2n$  variables and all coefficients are 0 or 1. Therefore, the maximal degree among the  $y_i$  is a primary measure of the complexity of this system, and hence of the APX algorithm. Each  $+$  operation performed by the algorithm increases the degree, while bitwise permutations do not increase it at all.

Algorithm	Internal state size	Addition bits	Sensitivity	Number of $+$ operations
Skein	256	64	9	278
Blake	256	32	10	1345
Cubehash	1024	32	13	6145

TABLE 2. Experimental results

More precisely, we may observe that if  $f$  and  $g$  are polynomial functions that represent single bits of output and  $\deg(f) \geq \deg(g)$  then

$$\deg(f +_m g) = m \deg(f) + \deg(g)$$

by the equation given in Proposition 3.1. Therefore, replacing  $+$  by  $+_m$  dramatically reduces the degrees of the encoding polynomials.

In principle, algorithms using Gröbner bases can be used to solve such systems of polynomial equations, see e.g. [4]. Neither the collision nor the preimage attacks we have outlined seem to be currently practical, although this could change due to an increase in computer power or more efficient Gröbner basis algorithms, an active area of research in mathematics.

Although length of time to find a Gröbner basis is difficult to predict, generally it is true that the higher the degree of the equations, the longer the algorithm will take, so the degree of a hash algorithm gives a good measure of algorithm complexity.

**Definition 3.3.** We define the **degree** of an APX hash function to be the maximum degree of its encoding polynomials.

For ARX algorithms, we have seen that this metric will be dominated by the number of times  $+$  is used in the algorithm.

**Definition 3.4.** Denote an ARX hash algorithm by  $H$ , and its output after hashing the message  $M$  by  $H(M)$ . Given an ARX hash algorithm  $H$ , let  $H_m$  denote the corresponding algorithm in which all instances of  $+$  have been replaced by  $+_m$ . We define the **sensitivity** of  $H$  to be the minimum  $m$  such that  $H_m(M) = H(M)$  for at least 0.1 percent of the inputs  $M$  of each fixed length.

The sensitivity measures how vulnerable a given algorithm would be to the types of attacks we have outlined above. Notice that the degree and the sensitivity are related because we would expect that an algorithm using  $k$  addition operations would have  $H_m(M) = H(M)$  with probability  $(\pi_m)^k$  by Corollary 2.7. This assumes that these operations occur independently and that the distribution of inputs to the addition operations are uniform.

#### 4. EXAMPLES FROM THE NIST COMPETITION

In this section, we use Monte Carlo experiments to estimate the sensitivity of some NIST competition algorithms [6]. We implemented versions of Blake [1] and Skein [3] that use truncated addition, and ran them using random inputs to determine how often these modified algorithms agree with the original algorithm. Cubehash [2] did not pass the second round of the NIST competition but also provides an interesting example for analysis. The results are displayed in Table 2.

These results were generated using 10 trials with 1,000,000 random inputs each. For these trials, the match between Skein using  $+_8$  and Skein using  $+$  was .001% while the match between Skein using  $+_9$  and Skein using  $+$  was .294%. The match between Blake using  $+_10$  and Blake using  $+$  was .106%. The match between Cubehash using  $+_13$  and Cubehash using  $+$  was 3.4319% whereas we found no matches at all between Cubehash using  $+_12$  and Cubehash using  $+$ .



These results show that we may replace  $+$  by the significantly simpler operation  $+_m$  (where  $m = 9, 10, \text{ or } 13$ ) and still achieve the same output at least 0.1% of the time. Therefore collisions found in the truncated addition versions of the algorithms would translate to collisions in the full algorithms a statistically useful percent of the time.

*Remark 4.1.* Since Blake uses 32-bit addition, our truncated approximation reduces the degree of each addition from 32 to degree 10. On the other hand, Skein uses 64-bit addition so our truncated approximation gives a much more dramatic reduction from degree 64 to degree 9. For this reason, we would say that Skein is the weaker algorithm.

*Remark 4.2.* There are a total of  $278 +$  operations in Skein. If all of the addition operations occurred in independently and in parallel, we would expect the probability of a match between Skein<sub>9</sub> (using  $+_9$ ) and Skein (using  $+$ ) to be  $(\pi_9(64))^{278} = (0.97387)^{278} = 0.000635732714225483$ . In our Monte Carlo experiment, we actually found matches with probability 0.00294.

While there are permutations included in each round that amount to the addition operations being in parallel, many of Skein's additions appear in series.

*Remark 4.3.* Blake has 1345 total additions and sensitivity 10, so we would expect Blake<sub>10</sub> to match Blake with probability  $(\pi_{10}(32))^{1345} = (0.99488)^{1345} = 0.0010036724$ . In our experiments, we actually found matches with probability 0.00106. This makes Blake almost perfectly efficient via our metric.

*Remark 4.4.* The corresponding results for Cubehash seem surprising. The program we used to compute the sensitivity of Cubehash used only 6145  $+$  operations. (The number of operations in Cubehash depends on the length of the message being hashed, so it is important to not use generic figures for this.)

We would expect Cubehash<sub>13</sub> to match Cubehash with probability  $(\pi_{13}(32))^{6145} = (0.99945)^{6145} = 0.0340243180867048$ . In our experiments, we actually found matches *less* often, with probability 0.00106.

To understand this result, note that differences between  $+$  and  $+_m$  arise from the addition of two numbers with long strings of 0/1 pairs in consecutive entries. If a hash algorithm were unlikely to turn inputs into their opposite entry and then add the result to the original, then it is plausible to have such a result. In fact, unlike the other hash algorithms, Cubehash uses only odd rotation constants which may make it less likely to generate such strings.

It would be interesting to understand the relationship between  $\pi_m(N)^{\text{number of } + \text{ operations}}$  and the experimental match percentages more precisely.

## 5. FUTURE WORK FOR Skein

The heart of Skein is the tweakable block cipher *Threefish*, and it is this cipher that we suggest analyzing using truncated addition. The basic structure of the *Threefish* cipher is four applications of a non-linear bijection (defined using add, rotate and xor operations) followed by the addition of a full-length subkey. More specifically, *Threefish* breaks the internal state of 256 bits into two pairs of 64-bit words and applies to each pair an ARX function called MIX. After this, the four words are permuted (the same permutation,  $\text{PERMUTE} = (0)(13)(2)$ , being used each time). The rotation constants internal to MIX are changed on a schedule for optimal dispersal, and a 'round' in *Threefish* is the application of one set of MIXs and one PERMUTE. Every four rounds, a 'subkey' of length 256 is added to the current state. The full specification of *Threefish* calls for 72 rounds, so 18 subkeys added in total.

Following the scheme outlined in Section 3, a single round of *Threefish* can be made to act on a set of variables

$$(x_0, \dots, x_{63}, y_0, \dots, y_{63}, z_0, \dots, z_{63}, w_0, \dots, w_{63}) = (\overline{x}, \overline{y}, \overline{z}, \overline{w}) = \mathbf{x}$$

producing 256 Boolean polynomials in the variables  $x_0, \dots, w_{63}$ , one polynomial for each coordinate. We call the  $i$ th such polynomial  $f_i$  and denote the full operation on all of these variables  $f = (f_0, f_1, \dots, f_{256})$ . We similarly define the polynomials  $f_i(x_0, \dots, w_{63})_m$  to be the coordinate functions for the truncated addition version of  $\text{Threefish}_m$  in which all ordinary additions are replaced by  $m$ -truncated addition.

Observe that  $f$  is a bijection. This is due to the fact that when any add, rotate or xor operation within MIX is applied to  $\bar{x}, \bar{y}$ , one of the two original inputs is retained. This follows from the definition

$$\text{MIX}(\bar{x}, \bar{y}) = (\bar{x} + \bar{y}, \rho(\bar{x}) \oplus (\bar{x} + \bar{y}))$$

where  $\rho$  is bitwise rotation.

We first consider the collision attack outlined in Section 3. Since there are no collisions if the step is a bijection, we must consider non-bijective rounds. As the non-bijectivity occurs from adding the subkey, the first interesting computation would be:

Let  $K_0$  be the first sub-key and  $K_1$  be the second. Let  $I$  be the ideal generated by

$$f(f(f(f(f(\mathbf{x} + K_0)_m)_m)_m)_m + K_1)_m - f(f(f(f(f(\mathbf{x}' + K_0)_m)_m)_m)_m + K_1)_m.$$

A Gröbner basis for this ideal would detect the interaction between two non-bijective rounds, yielding real information about the  $\text{Skein}_m$  algorithm. Although we were unable to reverse enough rounds of  $\text{Skein}_m$  to make a practical attack, we did reverse two rounds of the  $m = 2$  carry-approximated algorithm on 16-bits by computing a Gröbner basis<sup>1</sup>.

Next, we consider the preimage attack. A preimage attack has no restrictions on the number of rounds needed to be useful, as a preimage for even one round is often difficult. Let  $I$  be the ideal generated by

$$\mathbf{z} - f(\mathbf{x} + K_0)_m,$$

corresponding to the system of equations from Section 3. In order to solve for  $\mathbf{x}$  in terms of  $\mathbf{z}$  and produce a true inverse for one round of the algorithm with truncated addition, we will need to use a lex Gröbner basis algorithm (with the variables in  $\mathbf{z} < \mathbf{x}$ ) to produce an elimination ideal. As the rounds of  $\text{Threefish}_m$  are not identical (the rotation constants are different for each round), an inverse for two rounds would require the same analysis for the ideal generated by

$$\mathbf{z} - f(f(\mathbf{x} + K_0)_m)_m,$$

and, theoretically, this process could be carried out for all 72 rounds of  $\text{Threefish}_m$  where the rounds containing subkeys would force the introduction of additional variables. Although we do not have a practical attack, we were able to reverse three rounds of the  $m = 2$  carry-approximated algorithm on 12-bits by computing a Gröbner basis<sup>2</sup>.

We believe these approaches will lead to useful computations for others with more computing resources to explore.

<sup>1</sup>Using Sage/PolyBoRi on a 2.53 GHz Intel Core i5 MacBook Pro. We also investigated the  $m = 3$  carry-approximated algorithm on 24-bits for up to 3 rounds Skein. While the number of polynomials is always 24, and the degrees of these polynomials do not exceed 16, the maximum number of terms in each polynomial grows from 10 to 2521 to 236187 for 1, 2 and 3 rounds of Skein, respectively. We attempted to find a Gröbner basis for the ideal generated by these polynomials using Sage/PolyBoRi, Macaulay 2, and the Macaulay 2 package BooleanGB [4], but none of these returned results for 2 or more rounds. These computations would be more feasible if a parallel version of the Gröbner basis algorithm became available.

<sup>2</sup>Using Sage/PolyBoRi on a 2.53 GHz Intel Core i5 MacBook Pro. We were also able to reverse one round of the  $m = 2$  carry-approximated algorithm on 16-bits.



## 6. CONCLUSIONS

We have seen how to encode APX hash functions as systems of polynomials over  $\mathbb{F}_2$ . The degree of the approximation obtained by using  $m$ -truncated addition will be significantly smaller than the degree of the original APX function. The sensitivity measures how small we can let  $m$  be and still obtain a function that reasonably approximates original APX hash function.

One open question that arises from this work is how to construct differential attacks using the metrics we have described. It would also be interesting to examine the encoding polynomials for some of the NIST competition algorithms in detail, and compute Gröbner bases for them.

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